LEWY'S CURVES AND CHAINS ON
REAL HYPERSURFACES

BY

JAMES J. FARAN V

ABSTRACT. Lewy's curves on an analytic real hypersurface $M = \{r(z, \bar{z}) = 0\}$ in $\mathbf{C}^2$ are the intersections of M with any of the Segre hypersurfaces $Q_w = \{z: r(z, w) = 0\}$. If M is the standard unit sphere, these curves are chains in the sense of Chern and Moser. This paper shows the converse in the strictly pseudoconvex case: If all of Lewy's curves are chains, M is locally biholomorphically equivalent to the sphere. This is proven by analyzing the holomorphic structure of the space of chains. A similar statement is true about real hypersurfaces in $\mathbf{C}^n$, $n > 2$, in which case the proof relies on a pseudoconformal analogue to the theorem in Riemannian geometry which states that a manifold having "sufficiently many" totally geodesic submanifolds is projectively flat.

0. Introduction. Let $M = \{z \in \mathbf{C}^{n+1}: r(z, \bar{z}) = 0\}$ be a nonsingular $(dr \neq 0)$ real-analytic (r is a real-analytic function on $\mathbf{C}^{n+1}$) real ($r = \bar{r}$) hypersurface in $\mathbf{C}^{n+1}$. For example, M may be the boundary of a bounded domain D in $\mathbf{C}^{n+1}$. Since r is real-analytic, it makes sense to define a function $r(z, \zeta)$ on $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$, at least for ζ near $\bar{z}$. The set $\mathfrak{M} = \{(z, \zeta) \in \mathbf{C}^{n+1} \times \mathbf{C}^{n+1}: r(z, \zeta) = 0\}$ is the Segre family associated to M, and does not depend on the choice of the defining function r. $\mathfrak{M}$ is a family of complex hypersurfaces $Q_\zeta = \{z \in \mathbf{C}^{n+1}: r(z, \zeta) = 0\}$ parametrized by ζ. First studied by B. Segre [4], the Segre family associated to a real hypersurface has been the subject of some recent study ([1], [3]), most usefully in the biholomorphic classification of real ellipsoids [6].

As an example, let M be the unit sphere $S^{2n+1} = \{z \in \mathbf{C}^{n+1}: \sum_{\alpha=1}^{n+1}|z^\alpha|^2 - 1 = 0\}$. Then the associated Segre hypersurfaces $Q_\zeta = \{z \in \mathbf{C}^{n+1}: \sum_{\alpha=1}^{n+1} z^\alpha \bar{\zeta}_\alpha - 1 = 0\}$ are just the complex hyperplanes in $\mathbf{C}^{n+1}$.

Consider now the case $n = 1$–a real hypersurface M in $\mathbf{C}^2$. In this case, the associated Segre hypersurfaces Q_ζ will be complex curves, and if Q_ζ intersects M transversely, $Q_\zeta \cap M$ will be a real curve in M. As ζ varies, we obtain a family of curves in M. This family of curves was first brought to my attention by H. Lewy. In conversations with S. S. Chern and N. Stanton I have referred to these curves as Lewy's curves, and I see no reason for stopping now. (The theorem in §1 below I obtained from Professor Lewy–although the proof given is my own concoction.) Lewy's curves are those curves in M given by the intersection of M with one of its associated Segre hypersurfaces.

For $n > 1$, $Q_{\zeta_1} \cap \cdots \cap Q_{\zeta_n}$ will be, in general, a complex curve. So we can still refer to Lewy's curves, meaning the intersections $M \cap Q_{\zeta_1} \cap \cdots \cap Q_{\zeta_n}$.

Returning to our example above, it is easy to see that Lewy's curves on S^{2n+1} are precisely the intersections of S^{2n+1} with complex lines in $\mathbf{C}^{n+1}$. It is well known that these are chains in the sense of Chern and Moser [2]. One is naturally led to a question first posed by H. Lewy: Are Lewy's curves always chains? Or, more precisely, for what real hypersurfaces are Lewy's curves chains?

The answer is only the sphere. The purpose of this paper is to show that if M is a real hypersurface all of whose Lewy's curves are chains, then M is locally biholomorphically equivalent to the sphere S^{2n+1}. When $n = 1$, this is done by showing that the space of chains has a natural complex structure (which it does have if it is the space of Segre hypersurfaces) only when M is pseudoconformally flat (and hence locally biholomorphically equivalent to the sphere). When $n > 1$, this sort of argument gets complicated, and we resort to more geometric methods. We shall prove an analogue of the theorem in Riemannian geometry which states that if a Riemannian manifold M of dimension greater than two contains totally geodesic hypersurfaces tangent to every tangent hyperplane then M is projectively flat (and hence by Beltrami's theorem has constant curvature).

The outline of the paper is as follows. §1 contains the basic results needed about Segre families and Lewy's curves, §2 those about real hypersurfaces and chains. §3 takes care of the case $n = 1$, and §4 the case $n > 1$, including a proof of the analogue mentioned above.

Finally a word about indices: upper case latin $A, B, C, \ldots$ run from 0 to $n + 1$, lower case greek $\alpha, \beta, \gamma, \ldots$ from 1 to n, and lower case latin $j, k, l, \ldots$ from 1 to $n - 1$.

1. Segre families and Lewy's curves. Let $D = \{z \in \mathbf{C}^{n+1}: r(z, \bar{z}) < 0\}$ be a strictly pseudoconvex domain in $\mathbf{C}^{n+1}$ with nonsingular real-analytic boundary $M = \{z \in \mathbf{C}^{n+1}: r(z, \bar{z}) = 0\}$. Let

$$\mathfrak{M} = \{(z, \zeta) \in \mathbf{C}^{n+1} \times \mathbf{C}^{n+1}: r(z, \zeta) = 0\}$$

be the Segre family of hypersurfaces. $Q_\zeta = \{z \in \mathbf{C}^{n+1}: r(z, \zeta) = 0\}$ associated to M. In this section we shall state the basic facts about $\mathfrak{M}$ that we shall need. First, let us show that Lewy's curves actually exist.

THEOREM. *Let* $D = \{z \in \mathbf{C}^{n+1}: r(z, \bar{z}) < 0\}$ *be a strictly pseudoconvex domain in* $\mathbf{C}^{n+1}$ *with real-analytic boundary* $M = \{z \in \mathbf{C}^{n+1}: r(z, \bar{z}) = 0\}$, $Q_{\bar{w}} = \{z \in \mathbf{C}^{n+1}: r(z, \bar{w}) = 0\}$ *one of the associated Segre hypersurfaces. Then locally,*
  (i) *if* $w \in D$, $Q_{\bar{w}} \cap M = \varnothing$,
  (ii) *if* $W \in M = \partial D$, $Q_{\bar{w}} \cap M = \{w\}$,
  (iii) *if* $w \in \mathbf{C}^{n+1} \setminus \bar{D}$, $Q_w \cap M$ *is diffeomorphic to* S^{2n-1}.
(*By "locally" we mean the following: Let* $z_0 \in M$. *Then for some sufficiently small neighborhood* U *of* z_0 *in* $\mathbf{C}^{n+1}$ *we have*
  (i) *if* $w \in D \cap U$, $(Q_{\bar{w}} \cap M) \cap U = \varnothing$,
  (ii) *if* $w \in M \cap U$, $(Q_{\bar{w}} \cap M) \cap U = \{w\}$,
  (iii) *if* $w \in U \setminus (\bar{D} \cap U)$, $(Q_w \cap M) \cap U$ *is diffeomorphic to* S^{2n-1}.)

PROOF. For z in M near z_0 choose Moser normal coordinates for M centered at z in such a way that they depend smoothly on z. Define curves $\gamma_z(t)$ for $t \in (-\varepsilon, \varepsilon)$ by $\gamma_z(t) = (0, \ldots, C, it)$ in the normal coordinates centered at z. $\gamma_z(t)$ is a curve transverse to M passing through z. Define $\gamma: M \times (-\varepsilon, \varepsilon) \to \mathbf{C}^{n+1}$ by $\gamma(z, t) = \gamma_z(t)$. γ is a local diffeomorphism near $M \times \{0\}$. Thus any $w \in \mathbf{C}^{n+1}$ near z_0 is of the form $w = (0, \ldots, 0, it)$ in normal coordinates centered at some $z \in M$ near z_0. Thus it suffices to show that (i), (ii), and (iii) hold for M in normal form, $w = (0, \ldots, 0, it)$ while U depends continuously on the coefficients in the normal form.

In normal coordinates, the equation defining M is

$$\operatorname{Im}(z^{n+1}) = \sum_{\alpha, \beta = 1}^{n} g_{\alpha\bar\beta} z^{\alpha} z^{\bar\beta} + \sum_{\substack{K,L \\ |K|+|L| > 4 \\ |K|,|L| > 2}} F_{K\bar L}(\operatorname{Re}(z^{n+1})) Z^K Z^{\bar L}$$

where $g_{\alpha\bar\beta}$ is the identity matrix and K, L are multi-indices of dimension n (e.g., $K = (k_1, \ldots, k_n)$, $|K| = k_1 + \cdots + k_n$, $Z^K = (z^1)^{k_1} \cdots (z^n)^{k_n}$). If $w = (0, \ldots, 0, it)$, $Q_{\bar w}$ is given by $z^{n+1} = -it$. (This is the advantage of using the normal form—$Q_{\bar w}$ is just a hyperplane.) Therefore $Q_{\bar w} \cap M$ is given by

$$z^{n+1} = -it, \tag{1.1}$$

$$-t = \sum_{\alpha, \beta} g_{\alpha\bar\beta} z^{\alpha} z^{\bar\beta} + \sum_{K,L} F_{K\bar L}(0) Z^K Z^{\bar L}. \tag{1.2}$$

For small values of $|z^{\alpha}|$ the second term on the right-hand side of (1.2) is dominated by the first. Thus for small t and small $|z^{\alpha}|$, $Q_{\bar w} \cap M$ is diffeomorphic to the set given by

$$z^{n+1} = -it, \qquad -t = \sum_{\alpha, \beta} g_{\alpha\bar\beta} z^{\alpha} z^{\bar\beta}. \tag{1.3}$$

Moreover, the bounds on t and $|z^{\alpha}|$ may be chosen to depend continuously on the $F_{K\bar L}(0)$.

Examination of the equations (1.3) shows that $Q_{\bar w} \cap M$ is (i) $\varnothing$ if $t > 0$, (ii) the origin if $t = 0$, and (iii) diffeomorphic to the $(2n - 1)$-sphere in $\mathbf{C}^n$ if $t < 0$. But $t > 0 \, (= 0, < 0)$ precisely when w is inside (on the boundary of, outside) D.

This theorem is of greatest interest, perhaps, when $n = 1$. In that case, if $w \in \mathbf{C}^2 \setminus \bar D$, $Q_{\bar w} \cap M$ is a curve in M. These curves we call Lewy's curves. In higher dimensions ($n > 1$), $Q_{\bar w} \cap M$ will not have real dimension one. However, if we choose $w_1, \ldots, w_n$ so that the $Q_{\bar w_k}$ intersect transversely, $\bigcap_{k=1}^{n} Q_{\bar w_k} \cap M$ will be a curve, and we call such curves Lewy's curves.

For w_k near $z_0 \in M$ the transverse intersection of the $Q_{\bar w_k}$ is a generic condition. To see this, consider the following interpretation of $\mathfrak{M}$. $(z, \zeta) \in \mathfrak{M}$ if and only if $z \in Q_\zeta$. Thus to $(z, \zeta) \in \mathfrak{M}$ one can assign the hyperplane element $T_z Q_\zeta$—the tangent plane to Q_ζ at z. In this fashion we obtain a natural map $F: (z, \zeta) \to T_z Q_\zeta$ from $\mathfrak{M}$ to the space $\mathbf{P}T^*\mathbf{C}^{n+1}$ of hyperplane elements of $\mathbf{C}^{n+1}$. F is a holomorphic map of a complex manifold of dimension $2n + 1$ to one of the same dimension. A calculation of the Jacobian of F (see [3], cf. [6]) shows that F is a

local biholomorphism near points $(z, \bar{z}) \in \mathfrak{M}$ (since M is strictly pseudoconvex). Thus, in particular, F, as a map from $\{(z, \zeta) \in \mathfrak{M}: z = z_0\}$ to the projective n-space $\mathbf{P}T^*_{z_0}\mathbf{C}^{n+1}$ of complex hyperplanes in $\mathbf{C}^{n+1}$ through z_0, is a local biholomorphism. Now if $z_0 \in \bigcap_k Q_{\bar{w}_k}$, the intersection will be transverse at z_0 if the $T_{z_0}Q_{\bar{w}_k} = F(z_0, \bar{w}_k)$ are in general position (as points in the projective n-space $\mathbf{P}T^*_{z_0}\mathbf{C}^{n+1}$). This is well known to be a generic condition on the $T_{z_0}Q_{\bar{w}_k}$. Since F is a local biholomorphism, it is a generic condition on the w_k if the w_k are close enough to z_0.

Using the map F we can transfer any local structure on $\mathfrak{M}$ to one on $\mathbf{P}T^*\mathbf{C}^{n+1}$. The structure we have in mind is the following. Let $\mathcal{I}'$ be the C^∞ differential ideal on $\mathfrak{M}$ generated by $dz^1|_{\mathfrak{M}}, \ldots, dz^{n+1}|_{\mathfrak{M}}$, $\mathcal{I}''$ the C^∞ differential ideal on $\mathfrak{M}$ generated by $d\zeta_1|_{\mathfrak{M}}, \ldots, d\zeta_{n+1}|_{\mathfrak{M}}$. ($\mathcal{I}'$ ($\mathcal{I}''$) naturally corresponds to the ideal generated by forms of type $(1, 0)$ (type $(0, 1)$) restricted to M.) $\mathcal{I}'$ and $\mathcal{I}''$ are both closed ($d\mathcal{I}' \subset \mathcal{I}', d\mathcal{I}'' \subset \mathcal{I}''$) and generated by forms which are type $(1, 0)$ on the complex manifold $\mathfrak{M}$. $\mathcal{I}'$ and $\mathcal{I}''$ are invariant under holomorphic changes of coordinates in $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$ preserving the factors and give $\mathfrak{M}$ the structure of an abstract Segre family [3] from which all the local invariants of $\mathfrak{M}$ may be derived [1]. Using the map F we obtain similar ideals $\mathcal{I}'$, $\mathcal{I}''$ on the space of hyperplane elements $\mathbf{P}T^*\mathbf{C}^{n+1}$. Note that the Segre hypersurfaces are the leaves of the foliation $\mathcal{I}'' = 0$.

2. Real hypersurfaces and chains.
In this section we recall the definition of chains and construct a bundle over the real hypersurface M which corresponds to the bundle of line elements of $\mathbf{C}^{n+1}$. Chains correspond to leaves of a certain foliation of this bundle.

Let $\mathrm{SU}(n + 1, 1)$ be the group of complex $(n + 2) \times (n + 2)$ matrices A such that $AQ'\bar{A} = Q$, where

$$Q = \begin{bmatrix} 0 & 0 & i/2 \\ 0 & I_n & 0 \\ -i/2 & 0 & 0 \end{bmatrix}.$$

Let $\mathrm{su}(n + 1, 1)$ be the Lie algebra of $\mathrm{SU}(n + 1, 1)$ (matrices l such that $lQ + Q'\bar{l} = 0$). Elements of $\mathrm{SU}(n + 1, 1)$ act on $\mathbf{C}^{n+2}$ as linear transformations; hence they act on $\mathbf{P}^{n+1}$ as linear fractional transformations. This action preserves the real hyperquadric $\{[\zeta] \in \mathbf{P}^{n+1}: {}'\bar{\zeta}Q\zeta = 0\}$. $\mathrm{SU}(n + 1, 1)$ acts transitively on the real hyperquadric. The group H of elements of $\mathrm{SU}(n + 1, 1)$ fixing a given point of the real hyperquadric may be identified with the group of matrices

$$T = \begin{bmatrix} t & 0 & 0 \\ t_\alpha & t^\beta_\alpha & 0 \\ \tau & \tau^\beta & \bar{t}^{-1} \end{bmatrix} \in \mathrm{SU}(n + 1, 1). \tag{2.1}$$

Let M be a strictly pseudoconvex real hypersurface in $\mathbf{C}^{n+1}$. Then there is an intrinsically defined structure bundle Y over M (a principal bundle with group H) and an $\mathrm{su}(n + 1, 1)$-valued Cartan connection $\pi = (\pi^B_A)$ on Y (see [2]). Given a local section $\sigma: M \to Y$ one obtains an $\mathrm{su}(n + 1, 1)$-valued one-form $\tilde{\omega} = \sigma^*\pi$ on

M. Since any two sections differ by an action of H, since π is a Cartan connection any two such forms on M are related by

$$\tilde{\omega} = dT \cdot T^{-1} + T\tilde{\omega}T^{-1}, \tag{2.2}$$

where T is of the form (2.1).

π satisfies the structure equations

$$d\pi = \pi \wedge \pi + \Pi \tag{2.3}$$

where

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ \Pi^0_\alpha & \Pi^\beta_\alpha & 0 \\ \Pi^0_{n+1} & \Pi^\beta_{n+1} & 0 \end{bmatrix} \tag{2.4}$$

is a two-form with values in the Lie algebra of H. Moreover

$$\Pi^\beta_\alpha = \sum_{\rho,\sigma} S^\beta_{\alpha\rho\bar\sigma} \pi^\rho_0 \wedge \overline{\pi^\sigma_0} + \frac{1}{2} \sum_\rho V^\beta_{\alpha\rho} \pi^\rho_0 \wedge \pi^{n+1}_0$$

$$- \frac{1}{2} \sum_\sigma V^\beta_{\alpha\bar\sigma} \overline{\pi^\sigma_0} \wedge \pi^{n+1}_0,$$

$$\Pi^\beta_{n+1} = \frac{1}{2} \sum_{\rho,\sigma} V^\beta_{\rho\bar\sigma} \pi^\rho_0 \wedge \overline{\pi^\sigma_0} + \frac{1}{4} \sum_\rho P^\beta_\rho \pi^\rho_0 \wedge \pi^{n+1}_0$$

$$+ \frac{1}{4} \sum_\sigma Q^\beta_{\bar\sigma} \overline{\pi^\sigma_0} \wedge \pi^{n+1}_0,$$

$$\Pi^0_{n+1} = \frac{i}{2} \sum_{\rho,\sigma} P_{\rho\bar\sigma} \pi^\rho_0 \wedge \overline{\pi^\sigma_0} - \frac{1}{8} \sum_\rho R_\rho \pi^\rho_0 \wedge \pi^{n+1}_0$$

$$- \frac{1}{8} \sum_{\bar\sigma} R_{\bar\sigma} \overline{\pi^\sigma_0} \wedge \pi^{n+1}_0 \tag{2.5}$$

with

$$\sum_\alpha S^\alpha_{\alpha\rho\bar\sigma} = \sum_\alpha V^\alpha_{\alpha\rho} = \sum_\alpha \operatorname{Re}(P^\alpha_\alpha) = 0. \tag{2.6}$$

Π is the curvature of the connection π (again, see [2]).

Y is a bundle of coframes in the following sense. A section σ of Y (over an open set $U \subset M$) is determined by fixing a choice of $\tilde\omega^0_0 - \tilde\omega^{n+1}_{n+1} = \sigma^*(\pi^{n+1}_0)$, $\tilde\omega^\alpha_0 = \sigma^*(\pi^\alpha_0)$ and $\tilde\omega^0_0 - \tilde\omega^{n+1}_{n+1} = \sigma^*(\pi^0_0 - \pi^{n+1}_{n+1})$ (over U). One's choice of $\tilde\omega^{n+1}_0$ is determined up to a multiple, for $\tilde\omega^{n+1}_0$ annihilates the maximal complex tangent space $T_C M$ of M (the real hyperplane in $TM \subset TC^{n+1}$ preserved by the complex structure $J: TC^{n+1} \to TC^{n+1}$). One's choice of $\tilde\omega^\alpha_0$ is restricted by the condition that $\tilde\omega^{n+1}_0$, $\tilde\omega^1_0, \ldots, \tilde\omega^n_0$ form a basis for the differential forms on M that are the restriction of forms of type $(1, 0)$ on C^{n+1}. $\tilde\omega^0_0 - \tilde\omega^{n+1}_{n+1}$ can then be chosen to satisfy

$$d\tilde\omega^{n+1}_0 - \tilde\omega^\alpha_0 \wedge \tilde\omega^{n+1}_\alpha + \omega^{n+1}_0 \wedge \left(\omega^0_0 - \omega^{n+1}_{n+1}\right) = 0$$

(cf. (2.3)). The forms $\tilde\omega^{n+1}_0$, $\tilde\omega^\alpha_0$, $\tilde\omega^{n+1}_\alpha = 2i\overline{\tilde\omega^\alpha_0}$ form a basis for $T^*M \otimes C$.

The projections to M of the integral curves of the differential equation

$$\pi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

in Y are called chains. Chains are always transverse to the maximal complex tangent space of M. Moreover, given a point $p \in M$ and a tangent direction X transverse to the maximal complex tangent space at p there is a unique chain through p with tangent X. The differential equation (2.7) gives us more than just the curves—it also gives us an intrinsic projective parameter along each chain and a certain notion of parallel translation. If we wish to ignore these, we should look only at the differential equation

$$\pi_0^\alpha = \pi_\alpha^{n+1} = 0. \tag{2.8}$$

This is a system of second order differential equations on M. To make this more explicit, let γ be a curve transverse to the maximal complex tangent space of M. Let $\tilde\omega_0^{n+1}$, $\tilde\omega_0^\alpha$, $\tilde\omega_{n+1}^{n+1} - \tilde\omega_0^0$ be a local coframe on M, $\tilde\omega = (\tilde\omega_A^B)$ the corresponding connection form. Since γ is transverse to $\tilde\omega_0^{n+1} = 0$, we may parametrize $\gamma = \dot\gamma(t)$ so that $\tilde\omega_0^{n+1}(\gamma(T)) = 1$. Then if $\tilde\omega_0^{\beta^*} = \tilde\omega_0^\beta - \tilde\omega_0^\beta(\dot\gamma)\tilde\omega_0^{n+1}$, we have that $\tilde\omega_0^{\beta^*}(\dot\gamma) = 0$. So let $\tilde\omega^*$ be the connection form corresponding to the coframe $\tilde\omega_0^{n+1}$, $\tilde\omega_0^{\alpha^*}$, $\tilde\omega_{n+1}^{n+1} - \tilde\omega_0^0 - \sum_\beta \tilde\omega_\beta^{n+1}(\dot\gamma)\tilde\omega_0^\beta + \sum_\beta \tilde\omega_0^\beta(\dot\gamma)\tilde\omega_\beta^{n+1}$. Then γ is a chain if and only if $\tilde\omega_\beta^{n+1}(\dot\gamma) = 0$. Using (2.2) one calculates that

$$\tilde\omega_{n+1}^{\beta^*} = d\left(\tilde\omega_0^\beta(\dot\gamma)\right) + \sum_\alpha \tilde\omega_0^\alpha(\dot\gamma)\tilde\omega_\alpha^\beta - \sum_\rho \tilde\omega_0^\rho(\dot\gamma)\tilde\omega_0^\beta(\dot\gamma)\tilde\omega_\rho^{n+1}$$

$$+ \tilde\omega_{n+1}^\beta - \tilde\omega_0^\beta(\dot\gamma)\tilde\omega_{n+1}^{n+1}.$$

So the differential equation describing chains may be written

$$\frac{d}{dt}\left(\tilde\omega_0^\beta(\dot\gamma)\right) = \sum_\alpha \left[-\tilde\omega_\alpha^\beta(\dot\gamma) + \tilde\omega_0^\beta(\dot\gamma)\tilde\omega_\alpha^{n+1}(\dot\gamma) + \delta_\alpha^\beta\tilde\omega_{n+1}^{n+1}(\dot\gamma) \right]\tilde\omega_0^\alpha(\dot\gamma) - \tilde\omega_{n+1}^\beta(\dot\gamma).$$

$$\tag{2.9}$$

Our objective here is to study the "space of chains" (unparametrized, unframed chains). Unfortunately, because of the complicated global behavior of chains, the "space of chains" is quite intractable. To avoid this, we consider instead the space of pointed chains, an element of which is a chain γ together with a point p on γ. Equivalently, we may take the point p together with a direction at p transverse to the maximal complex tangent space (for such a direction determines a unique chain γ).

So let $E = \{X \in TM \setminus T_\mathbf{C}M\}/X \sim \lambda X$, $\lambda \in \mathbf{R}$. Given $X_0 \in M_p \setminus T_\mathbf{C}M_p$, any other $X \in TM_p \setminus T_\mathbf{C}M_p$ may be written uniquely as $X = \tilde Y + \lambda X_0$, for $\lambda \in \mathbf{R}$, $\tilde Y \in T_\mathbf{C}M_p$. Or, alternatively, $X = \lambda(Y + X_0)$, $\lambda Y = \tilde Y$. Thus one may associate to the direction $[X] \in E_p$ the vector $Y \in T_\mathbf{C}M_p$. Thus E_p may be identified with $T_\mathbf{C}M_p$ (the identification depending on the choice of X_0). E is a fiber bundle over M with fiber $\mathbf{C}^n$.

E may also be viewed as a quotient bundle of Y. To a direction $[X]$ we can associate all coframes $\tilde\omega_0^{n+1}$, $\tilde\omega_0^\alpha$, $\tilde\omega_{n+1}^{n+1} - \tilde\omega_0^0$ with $\tilde\omega_0^\alpha(X) = 0$. The set of such

coframes is acted on transitively by the group H' of matrices

$$T = \begin{bmatrix} t & 0 & 0 \\ 0 & t_\alpha^\beta & 0 \\ \tau & 0 & \bar{t}^{-1} \end{bmatrix} \in H. \tag{2.10}$$

In this fashion, we obtain a mapping $E \to Y/H'$.

A third way of looking at E is as a real hypersurface in the bundle of line elements over $\mathbf{C}^{n+1}$. To $[X] \in E_p$ associate the complex line in $\mathbf{C}^{n+1}$ through p spanned by X.

One can put coordinates on E in the following way. Let $p \in M$. Choose a basis $\omega^1, \ldots, \omega^{n+1}$ for $T^{1,0}\mathbf{C}^{n+1}$ in a neighborhood of p so that $\omega^{n+1}|_M$ annihilates $T_\mathbf{C}M$. Then choose a coframe $\tilde{\omega}_0^{n+1} = \omega^{n+1}|_M$, $\tilde{\omega}_0^\alpha = \omega^\alpha$, $\tilde{\omega}_{n+1}^{n+1} - \tilde{\omega}_0^0$ for M. Then if $(z^1, \ldots, z^{n+1})$ are coordinates on $\mathbf{C}^{n+1}$, we may put coordinates $(z^1, \ldots, z^{n+1}, \zeta^1, \ldots, \zeta^n)$ on $\mathbf{P}T\mathbf{C}^{n+1}$ (the bundle of line elements over $\mathbf{C}^{n+1}$) by associating to $(z^1, \ldots, z^{n+1}, \zeta^1, \ldots, \zeta^n)$ the tangent line at $(z^1, \ldots, z^{n+1})$ given by $\omega^\alpha + \zeta^\alpha \omega^{n+1} = 0$. If $(z^1, \ldots, z^{n+1}) \in M$, we similarly have coordinates on E: to $(z^1, \ldots, z^{n+1}, \zeta^1, \ldots, \zeta^n)$ associate the tangent line given by $\tilde{\omega}_0^\alpha + \zeta^\alpha \tilde{\omega}_0^{n+1} = 0$. With respect to these coordinates the map $E \to \mathbf{P}T\mathbf{C}^{n+1}$ is the identity map.

Having chosen the coframe $\tilde{\omega}_0^{n+1}$, $\tilde{\omega}_0^\alpha$, $\tilde{\omega}_{n+1}^{n+1} - \tilde{\omega}_0^0$ locally, we also get coordinates for Y. The choice of coframe gives us a section $\sigma: M \to Y$. Let $\tilde{\omega} = \sigma^*\pi$. Then to a point $(x, T) \in M \times H$ associate the coframe $\tilde{\omega}_0^{n+1*}$, $\tilde{\omega}_0^{\alpha*}$, $\tilde{\omega}_{n+1}^{n+1*} - \tilde{\omega}_0^{0*}$ at x, where $\tilde{\omega}^* = T\tilde{\omega}T^{-1}$. This gives coordinates (x, T) on Y. In these coordinates, the connection on Y is given by

$$\pi = dT \cdot T^{-1} + T\tilde{\omega}T^{-1}. \tag{2.11}$$

Similarly, if we write

$$\mathbf{C}^n \cong \left\{ T = \begin{bmatrix} 1 & 0 & 0 \\ t_\alpha & \delta_\alpha^\beta & 0 \\ \tau & \tau^\beta & 1 \end{bmatrix} \in H, \tau^\beta \in \mathbf{C}, \operatorname{Im}(\tau) = 0 \right\}, \tag{2.12}$$

E is coordinatized by sending $(x, T) \in M \times \mathbf{C}^n$ to the direction $\tilde{\omega}_0^{\alpha*} = 0$, where $\tilde{\omega}^* = T\tilde{\omega}T^{-1}$. Then a section $\sigma: E \to Y$ is determined by a matrix $S \in H'$: $\sigma(x, T) = (x, S \cdot T)$. From (2.11) it follows that

$$\theta := \sigma^*\pi = dS \cdot S^{-1} + S(T\tilde{\omega}T^{-1} + dT \cdot T^{-1})S^{-1}. \tag{2.13}$$

The forms θ_0^{n+1}, θ_0^α, $\overline{\theta_0^\alpha}$, θ_α^{n+1}, $\overline{\theta_\alpha^{n+1}}$ span $T^*F \otimes \mathbf{C}$ and are well defined up to a transformation

$$\theta_0^{n+1*} = |t|^{-2}\theta_0^{n+1}, \quad \theta_0^{\alpha*} = \sum_\beta t^{-1}t_\beta^\alpha\theta_0^\beta,$$

$$\theta_{n+1}^{\alpha*} = \sum_\beta -\bar{t}\,t^{-1}\tau t_\beta^\alpha\theta_0^\beta + \sum_\beta \bar{t}\,t_\beta^\alpha\theta_{n+1}^\beta, \tag{2.14}$$

where

$$S = \begin{bmatrix} t & 0 & 0 \\ 0 & t_\alpha^\beta & 0 \\ \tau & 0 & \bar{t}^{-1} \end{bmatrix} \in H'. \tag{2.15}$$

In particular, the equations $\theta_0^\alpha = \overline{\theta_0^\alpha} = \theta_{n+1}^\alpha = \overline{\theta_{n+1}^\alpha} = 0$ give a well-defined foliation of F by curves. It follows from the definition of chains that the projection of these curves to M are chains of M. Moreover, if $\gamma(t)$ is a chain on M, the curve $[\dot{\gamma}(t)]$ in E will be an integral curve of the foliation (2.15).

3. $n = 1$.

THEOREM. *Let M be a strictly pseudoconvex real-analytic real hypersurface in $\mathbf{C}^2$. If all of Lewy's curves are chains, M is locally biholomorphically equivalent to the sphere $\{(z, w): |z|^2 + |w|^2 = 1\}$.*

The proof of this theorem can be reduced to an examination of the structure equations (2.3). Because we are in $\mathbf{C}^2$, a tangent hyperplane is a tangent line: $\mathbf{P}T^*\mathbf{C}^2 \cong \mathbf{P}T\mathbf{C}^2$. On $\mathbf{P}T^*\mathbf{C}^2$ we have the Segre family structure associated to M, in particular, the ideal $\mathcal{I}$, the integral curves of which are the Segre hypersurfaces. Restricting the bundle $\mathbf{P}T^*\mathbf{C}^2 \cong \mathbf{P}T\mathbf{C}^2$ to M we obtain E. The integral curves of $\mathcal{I}|E$ projected to M are Lewy's curves. Our assumption is that the integral curves of $\mathcal{I}|E$ are all chains, hence integral curves of (2.15).

Suppose $\theta_0^\alpha = \theta_0^1$ and $\theta_{n+1}^\alpha = \theta_2^1$ are the restrictions to E of forms of type $(1, 0)$ on $\mathbf{P}T\mathbf{C}^2$. Then $\mathcal{I}|E$ is generated by θ_0^1 and θ_2^1. The structure equations (2.3) then give

$$d\theta_2^1 = \theta_2^1 \wedge (\theta_1^1 - \theta_2^2) + \theta_2^0 \wedge \theta_0^2 + \Theta_2^1,$$

where $\theta = \sigma^*\pi$, $\Theta = \sigma^*\Pi$, for some section $\sigma: F \to Y$. From (2.5), $\Theta_2^1 = \frac{1}{4}\sigma^*(Q_1^1)\overline{\theta_0^1} \wedge \theta_0^2$. Thus, modulo (θ_0^1, θ_2^1), $d\theta_2^1 \equiv \frac{1}{4}\sigma^*(Q_1^1)\overline{\theta_0^1} \wedge \theta_0^2$. Since $\overline{\theta_0^1}$ and θ_0^2 are linearly independent, $d\mathcal{I} \subset \mathcal{I}$ implies $\sigma^*(Q_1^1) = 0$. Hence Q_1^1 is identically zero. It then follows from Bianchi identities [5] that the curvature π vanishes identically, i.e., M is flat. A standard application of the Frobenius theorem yields that M is locally biholomorphically equivalent to the sphere.

We have left to show that our supposition above is justified.

LEMMA. *The forms θ_0^α, θ_{n+1}^α are the restriction to $E \subset \mathbf{P}T\mathbf{C}^{n+1}$ of forms of type $(1, 0)$.*

PROOF. Choose a coframe $\tilde{\omega}_0^{n+1}$, $\tilde{\omega}_0^\alpha$, $\tilde{\omega}_{n+1}^{n+1} - \tilde{\omega}_0^0$ for M. We have seen that $\tilde{\omega}_0^{n+1}$ and $\tilde{\omega}_0^\alpha$ are restrictions of forms of type $(1, 0)$. Taking $S =$ identity matrix in the definition (2.13) of θ, we obtain $\theta_0^\alpha = \tilde{\omega}_0^\alpha - \tau^\alpha\tilde{\omega}_0^{n+1}$, where the τ^α are the fiber coordinates (2.12) of $E \to M$. So the θ_0^α are the restrictions of forms of type $(1, 0)$. To check that the θ_{n+1}^α are the restrictions of forms of type $(1, 0)$ it suffices to check that for any local holomorphic section $\sigma: \mathbf{C}^{n+1} \to \mathbf{P}T\mathbf{C}^{n+1}$, $(\sigma|_M)^*\theta_{n+1}^\alpha$ is the restriction of forms of type $(1, 0)$, or, alternatively, that

$$(\sigma|_M)^*\theta_{n+1}^\alpha \equiv 0 \quad \mathrm{mod}(\tilde{\omega}_0^{n+1}, \tilde{\omega}_0^\alpha).$$

So let $\sigma: \mathbf{C}^{n+1} \to \mathbf{P}T\mathbf{C}^{n+1}$ be a local holomorphic section. Choose forms ω^α (of type $(1, 0)$) so that, for $z \in \mathbf{C}^{n+1}$, the line $\sigma(z)$ is given by $\omega^\alpha(z) = 0$. σ is holomorphic if and only if $d\omega^\alpha \equiv 0 \mod(\omega^1, \ldots, \omega^n)$. Choose a local coframe $\tilde{\omega}_0^{n+1*}$, $\tilde{\omega}_0^{\alpha*}$, $\tilde{\omega}_{n+1}^{n+1*} - \tilde{\omega}_0^{0*}$ for M with $\tilde{\omega}_0^{\alpha*} = \omega^\alpha|_M$ and let $\tilde{\omega}$ be the corresponding

connection form. Then, in the coordinates (2.12), $\sigma|_M$ is given by $x \mapsto (x, I)$, where I is the $(n + 2) \times (n + 2)$ identity matrix. Thus $(\sigma|_M)^* \theta_{n+1}^{\alpha*} = \tilde{\omega}_{n+1}^{\alpha*}$, where $\theta^* = T\tilde{\omega}^* T^{-1} + dT \cdot T^{-1}$ as in (2.13). The structure equations tell us that, modulo (ω^β), $d\omega^\alpha \equiv d\tilde{\omega}_0^{\alpha*} \equiv \tilde{\omega}_0^{n+1*} \wedge \tilde{\omega}_{n+1}^{\alpha*}$. Thus $d\omega^\alpha \equiv 0 \mod(\omega^\beta)$ implies that $\tilde{\omega}_{n+1}^{\alpha*} \equiv 0$ $\mod(\tilde{\omega}_0^{n+1}, \tilde{\omega}_0^\beta)$. The lemma then follows from the transformation law (2.14).

4. $n > 1$. When $n > 1$, $\mathbf{P}T^*\mathbf{C}^{n+1}$ is no longer $\mathbf{P}T\mathbf{C}^{n+1}$; hyperplanes are not lines. So the proof given in the previous section breaks down. It may be possible to patch it up–the definition of Lewy's curves, involving as it does certain generic intersections, is slightly jury-rigged, and so one might expect to need a bit of jury-rigging to prove anything. We shall instead obtain the same theorem as before (but for $n > 1$) by proving an analogue of a classical theorem in Riemannian geometry: Any Riemannian manifold of dimension greater than two with the property that through any point and tangent to any tangent hyperplane there exists a totally geodesic hypersurface is projectively flat (and hence has constant curvature).

Our first step is to describe what is meant by a totally geodesic hypersurface in a real hypersurface M. Because M has a partial complex structure ($T_{\mathbf{C}}M$), we shall not look at all hypersurfaces in M, but rather those which inherit some of the complex structure.

DEFINITION. $\Sigma \subset TM_p$ is a *tangent CR hyperplane* if (1) Σ is a real codimension two hyperplane in TM_p, (2) Σ is transverse to the maximal complex tangent space $T_{\mathbf{C}}M_p$, and (3) $\Sigma \cap T_{\mathbf{C}}M_p$ is a complex hyperplane in $T_{\mathbf{C}}M_p$ (i.e., is preserved by the complex structure J).

DEFINITION. A real codimension two submanifold $S \subset M$ is a *CR hypersurface* in M if every tagent plane TS_p is a tangent CR hyperplane.

If S is a real-analytic CR hypersurface in M one can show that S is the transverse intersection of M and a complex analytic hypersurface in $\mathbf{C}^{n+1}$.

Let S be a CR hypersurface in a real hypersurface M. One can choose a coframe $\tilde{\omega}_0^{n+1}, \tilde{\omega}_0^\alpha, \tilde{\omega}_{n+1}^{n+1} - \tilde{\omega}_0^0$ on M so that S is given by

$$\tilde{\omega}_0^n = 0, \tag{4.1}$$

since $TS \cap T_{\mathbf{C}}M$ is complex. The Frobenius integrability conditions are

$$d\tilde{\omega}_0^n \equiv 0 \quad \mod(\tilde{\omega}_0^n, \overline{\tilde{\omega}_0^n}). \tag{4.2}$$

Thus, from the structure equations

$$\sum_j \tilde{\omega}_0^j \wedge \tilde{\omega}_j^n + \tilde{\omega}_0^{n+1} \wedge \omega_{n+1}^n \equiv 0 \quad \mod(\tilde{\omega}_0^n, \overline{\tilde{\omega}_0^n}). \tag{4.3}$$

(Recall $j, k, l, \ldots = 1, \ldots, n - 1$.) Therefore, applying Cartan's lemma to (4.3),

$$\tilde{\omega}_j^n \equiv \sum_k A_{jk}\tilde{\omega}_0^k + B_j\tilde{\omega}_0^{n+1} \quad \mod(\tilde{\omega}_0^n, \overline{\tilde{\omega}_0^n}),$$

$$\tilde{\omega}_{n+1}^n \equiv \sum_k B_{jk}\tilde{\omega}_0^k + C\tilde{\omega}_0^{n+1} \quad \mod(\tilde{\omega}_0^n, \overline{\tilde{\omega}_0^n}) \tag{4.4}$$

for some functions A_{jk}, B_j, C with

$$A_{jk} = A_{kj}. \tag{4.5}$$

DEFINITION. The quadratic form

$$\text{II} = \sum_{j,k} A_{jk}\tilde{\omega}_0^j\tilde{\omega}_0^k + 2\sum_k B_k\tilde{\omega}_0^k\tilde{\omega}_0^{n+1} + C(\tilde{\omega}_0^{n+1})^2 \tag{4.6}$$

is the second fundamental form of the CR hypersurface S.

PROPOSITION (4.7). $\text{II} \equiv 0$ *if and only if every chain in M tangent to S is contained in S.*

PROOF. First, assume every chain in M tangent to S is contained in S. Choose a local coframe $\tilde{\omega}_0^{n+1}$, $\tilde{\omega}_0^\alpha$, $\tilde{\omega}_{n+1}^{n+1} - \tilde{\omega}_0^0$ for M with $S = \{\tilde{\omega}_0^n = 0\}$. Let $\gamma(t)$ be a chain of M contained in S, parametrized so $\tilde{\omega}_0^{n+1}(\dot\gamma) = 1$. By assumption, $\tilde{\omega}_0^n(\dot\gamma) = 0$. Thus, from (2.9),

$$0 = \frac{d}{dt}(\tilde{\omega}_0^n(\dot\gamma)) = -\sum_j \tilde{\omega}_j^n(\dot\gamma)\tilde{\omega}_0^j(\dot\gamma) - \tilde{\omega}_{n+1}^n(\dot\gamma)$$

$$= -\sum_j \tilde{\omega}_j^n(\dot\gamma)\tilde{\omega}_0^j(\dot\gamma) - \tilde{\omega}_{n+1}^n(\dot\gamma)\tilde{\omega}_0^{n+1}(\dot\gamma) = -\text{II}(\dot\gamma, \dot\gamma).$$

This holds for any $\dot\gamma \in TS$ with $\tilde{\omega}_0^{n+1}(\dot\gamma) = 1$, hence for any $\dot\gamma \in TS$ with $\tilde{\omega}_0^{n+1}(\dot\gamma) \neq 0$. By continuity, it holds for all $\dot\gamma \in TS$, i.e., $\text{II} \equiv 0$.

Conversely, if $\text{II} \equiv 0$, the above calculations show that the chain equations (2.9) reduce to a system of ordinary differential equations on S. It follows that every chain in M tangent to S is contained in S.

DEFINITION. A CR hypersurface is totally geodesic if the associated second fundamental form $\text{II} \equiv 0$.

Suppose $S = \{\tilde{\omega}_0^n = 0\}$ is totally geodesic, so $\tilde{\omega}_j^n \equiv \tilde{\omega}_{n+1}^n \equiv 0 \bmod(\tilde{\omega}_0^n, \tilde{\omega}_0^{\bar n})$ (and hence, by the symmetries of $\tilde{\omega}$, $\tilde{\omega}_n^j \equiv \tilde{\omega}_n^{\overline{n+1}} \equiv 0 \bmod(\tilde{\omega}_0^n, \tilde{\omega}_0^{\bar n})$). From the structure equations (2.3) it follows that, $\bmod(\tilde{\omega}_0^n, \tilde{\omega}_0^{\bar n})$, $0 \equiv d\tilde{\omega}_j^n \equiv \Pi_j^n$, where $\Pi = d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega}$ is the curvature of the connection form $\tilde{\omega}$. Writing

$$\Pi_j^n = \sum_{\alpha,\sigma} S_{j\alpha\bar\sigma}^n\tilde{\omega}_0^\alpha \wedge \tilde{\omega}_0^{\bar\sigma} + \frac{1}{2}\sum_\alpha V_{j\alpha}^n\tilde{\omega}_0^\alpha \wedge \tilde{\omega}_0^{n+1}$$

$$- \frac{1}{2}\sum_\sigma V_{j\bar\sigma}^n\tilde{\omega}_0^{\bar\sigma} \wedge \tilde{\omega}_0^{n+1}, \tag{4.8}$$

we have, in particular,

$$S_{jk\bar l}^n = 0. \tag{4.9}$$

So the existence of a totally geodesic CR hypersurface implies that a certain part of the curvature must vanish. If we have many totally geodesic CR hypersurfaces, we would expect all of the curvature to vanish. Indeed, we have the following.

THEOREM. *Let M be a real-analytic strictly pseudoconvex real hypersurface in $\mathbf{C}^{n+1}$, $n \geqslant 2$. Let X be a vector field on M transverse to the maximal complex tangent space $T_C M$. If, for all p in M and for all tangent CR hyperplanes $\Sigma \subset TM_p$ containing $X(p)$, there exists a totally geodesic CR hypersurface S tangent to Σ at p, then M is flat, i.e., its curvature $\Pi = 0$.*

REMARKS. (1) M is flat if and only if M is locally biholomorphically equivalent to the sphere $S^{2n+1} = \{z \in \mathbf{C}^{n+1}: \|z\| = 1\}$. (2) The proof that follows actually proves a version of the theorem that is stronger in two ways. First, M need not be analytic, indeed we just need M to be a smooth nondegenerate Cauchy-Riemann manifold. Second, we do not need quite so many totally geodesic CR hypersurfaces. The space of tangent CR hyperplanes Σ containing X is a bundle over M with fiber complex projective space $\mathbf{P}^n$. Indeed, if we split $T_\mathbf{C}M \otimes \mathbf{C} = H^{1,0} \oplus H^{0,1}$, where $H^{1,0}$ ($H^{0,1}$) is the plus (minus) i eigenspace of the complex structure J, we can identify the space of CR hypersurfaces with the projectivization $\mathbf{P}H^{1,0*}$ of the dual of $H^{1,0}$. (An element of $\mathbf{P}H^{1,0*}$ is a line in $H^{1,0*}$; the annihilator of this line is a hyperplane T in $H^{1,0}$; $T \oplus \bar{T} \subset TM \otimes \mathbf{C}$ is preserved under conjugation, so $T \oplus \bar{T} = T_\mathbf{R} \otimes \mathbf{C}$ for some hyperplane $T_\mathbf{R}$ in $T_\mathbf{C}M$; then Σ is the span of $T_\mathbf{R}$ and X.) The theorem above remains true if we merely assume that there exists an open set U in $\mathbf{P}H^{1,0*}$ whose projection to M is all of M such that for all Σ in U there is a totally geodesic hypersurface tangent to Σ. We shall need to use this stronger result later. (3) This theorem is local. If the hypotheses are true in a neighborhood of a point q in M, the conclusion is true in a neighborhood of q.

PROOF OF THEOREM. Choose a local coframe on M (hence a connection form $\tilde{\omega}$) so that $\tilde{\omega}_0^\alpha(X) = 0$. Let $\Pi = d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega}$ be the corresponding curvature form. Π may be written in a form similar to (2.5) (cf. (4.8)). We have seen that if $\tilde{\omega}_0^n = 0$ defines a totally geodesic CR hypersurface, then

$$\sum_{\alpha,\sigma} S_{j\alpha\bar{\sigma}}^n \tilde{\omega}_0^\alpha \wedge \tilde{\omega}_0^{\bar{\sigma}} \equiv 0 \quad \mathrm{mod}\left(\tilde{\omega}_0^n, \overline{\tilde{\omega}_0^n}\right).$$

If we eliminate the dependence on the special index n, we obtain the statement that if $u_\alpha \tilde{\omega}_0^\alpha = 0$ defines a totally geodesic submanifold, then

$$\sum_{\alpha,\bar{\sigma},\gamma,\bar{\sigma}} v^\beta u_\gamma S_{\beta\alpha\bar{\sigma}}^\gamma \tilde{\omega}_0^\alpha \wedge \overline{\tilde{\omega}_0^\sigma} \equiv 0 \quad \mathrm{mod}\left(\sum_\alpha u_\alpha \tilde{\omega}_0^\alpha, \sum_\alpha \overline{u_\alpha}\ \overline{\tilde{\omega}_0^\alpha}\right) \qquad (4.10)$$

for all v^β such that

$$\sum_\beta v^\beta u_\beta = 0. \qquad (4.11)$$

Our assumption is that this statement is true at all points of M and for all vectors $(u_1, \ldots, u_n)$. In particular, consider vectors of the form $(u_1, \ldots, u_{n-1}, -1)$. (4.12) then becomes

$$v^n = \sum_j v^j u_j. \qquad (4.12)$$

Thus the v^j may be chosen arbitrarily, at which point v^n is fixed. Modulo $(\sum_\alpha u_\alpha \tilde{\omega}_0^\alpha, \sum_\alpha \overline{u_\alpha}\ \overline{\tilde{\omega}_0^\alpha})$ we have

$$\tilde{\omega}_0^n \equiv \sum_j u_j \tilde{\omega}_0^j, \qquad \overline{\tilde{\omega}_0^n} \equiv \sum_j \overline{u_j}\ \overline{\tilde{\omega}_0^j}. \qquad (4.13)$$

Using (4.12) and (4.13), (4.10) becomes

$$\sum_{l,m} \sum_j v^j \left\{ \sum_k u_k S^k_{jl\overline{m}} + u_k u_l S^k_{jn\overline{m}} + u_k \overline{u_m} S^k_{jl\overline{n}} + u_k u_l u_m S^k_{jn\overline{n}} \right.$$

$$+ u_j u_k S^k_{nl\overline{m}} + u_j u_k u_l S^k_{nn\overline{m}} + u_j u_k \overline{u_m} S^k_{nl\overline{n}} + u_j u_k u_l \overline{u_m} S^k_{nn\overline{n}}$$

$$- \left[S^n_{jl\overline{m}} + u_l S^n_{jn\overline{m}} + \overline{u_m} S^n_{jl\overline{n}} + u_l \overline{u_m} S^n_{jn\overline{n}} + u_j S^n_{nl\overline{m}} + u_j u_l S^n_{nn\overline{m}} \right.$$

$$\left. \left. + u_j \overline{u_m} S^n_{nl\overline{n}} + u_j u_l \overline{u_m} S^n_{nn\overline{n}} \right] \right\} \tilde{\omega}^l_0 \wedge \overline{\tilde{\omega}^m_0}$$

$$= 0.$$

Since the v^j are arbitrary, and because the $\tilde{\omega}^l_0$ and $\overline{\tilde{\omega}^m_0}$ are linearly independent, the term in the curly brackets must vanish. That term is a polynomial in u_j and $\overline{u_j}$. Our hypothesis is that this polynomial vanishes identically. (The weaker hypothesis mentioned above is that the polynomial vanishes on an open set of the u_j–but in that case it must also vanish identically.) Thus all the coefficients must vanish. It remains to show that this implies that the tensor $S^\gamma_{\alpha\beta\overline{\sigma}} = 0$, because if $S^\gamma_{\alpha\beta\overline{\sigma}} = 0$, it follows from the Bianchi identities [5] that $\Pi = 0$.

After symmetrizing where necessary, the vanishing of the coefficients above gives us that for all $j, k, l, m, p = 1, \ldots, n - 1$

$$S^n_{jl\overline{m}} = 0, \tag{4.14}$$

$$S^n_{jl\overline{n}} = 0, \tag{4.15}$$

$$S^k_{jl\overline{m}} - \delta^k_l S^n_{jn\overline{m}} - \delta^k_j S^n_{ln\overline{m}} = 0, \tag{4.16}$$

$$S^k_{jl\overline{n}} - \delta^k_l S^n_{jn\overline{n}} - \delta^k_j S^n_{ln\overline{n}} = 0, \tag{4.17}$$

$$\delta^p_l S^k_{jn\overline{m}} + \delta^p_j S^k_{ln\overline{m}} - \left(\delta^k_j \delta^p_l + \delta^k_l \delta^p_j \right) S^n_{nn\overline{m}} = 0, \tag{4.18}$$

$$\delta^p_l S^k_{jn\overline{n}} + \delta^p_j S^k_{ln\overline{n}} - \left(\delta^k_j \delta^p_l + \delta^k_l \delta^p_j \right) S^n_{nn\overline{n}} = 0, \tag{4.19}$$

$$S^k_{nn\overline{m}} = 0, \tag{4.20}$$

$$S^k_{nn\overline{n}} = 0. \tag{4.21}$$

Summing (4.18) and (4.19) over $l = p = 1, \ldots, n - 1$,

$$n S^k_{jn\overline{m}} - n \delta^k_j S^n_{nn\overline{m}} = 0, \tag{4.22}$$

$$n S^k_{jn\overline{n}} - n \delta^k_j S^n_{nn\overline{n}} = 0. \tag{4.23}$$

Summing (4.16), (4.17), (4.22), (4.23), over $j = k = 1, \ldots, n - 1$,

$$\sum_k S^k_{kl\overline{m}} - n S^n_{ln\overline{m}} = 0, \qquad \sum_k S^k_{kl\overline{n}} - n S^n_{ln\overline{n}} = 0,$$

$$\sum_k S^k_{kn\overline{m}} - n S^n_{nn\overline{m}} = 0, \qquad \sum_k S^k_{kn\overline{n}} - n S^n_{nn\overline{n}} = 0. \tag{4.24}$$

The trace conditions (2.6) give that $\sum_k S^k_{k\alpha\overline{\beta}} = -S^n_{n\alpha\overline{\beta}}$. Using this in (4.24) we obtain $-(n + 1)S^n_{ln\overline{m}} = 0$, $-(n + 1)S^n_{ln\overline{n}} = 0$, $-(n + 1)S^n_{nn\overline{m}} = 0$, $-(n + 1)S^n_{nn\overline{n}} = 0$. Therefore

$$S^n_{ln\overline{m}} = 0, \qquad S^n_{ln\overline{n}} = 0,$$

$$S^n_{nn\overline{m}} = 0, \qquad S^n_{nn\overline{n}} = 0. \tag{4.25}$$

Substituting (4.25) into (4.16), (4.17), (4.22) and (4.23) we obtain

$$S^l_{jk\bar{m}} = 0, \qquad S^l_{jk\bar{n}} = 0,$$

$$S^k_{jn\bar{m}} = 0, \qquad S^k_{jn\bar{n}} = 0. \tag{4.26}$$

Examination of (4.14), (4.15), (4.20), (4.21), (4.25) and (4.26) and use of the symmetry $S^\gamma_{\alpha\beta\bar{\sigma}} = S^\gamma_{\beta\alpha\bar{\sigma}}$ (which follows from the Bianchi identity $0 = d^2\tilde{\omega}^\alpha_0 = \tilde{\omega}^\beta_0 \wedge \Pi^\alpha_\beta + \tilde{\omega}^{n+1}_0 \wedge \Pi^\alpha_{n+1}$) shows that $S^\gamma_{\alpha\beta\bar{\sigma}} = 0$ for all α, β, γ, σ, finishing the proof of the theorem.

Let us return now to the consideration of Lewy's curves. Let M be a strictly pseudoconvex real-analytic real hypersurface in $\mathbf{C}^{n+1}$, $n \geqslant 2$. Recall that in this case, Lewy's curves are defined as $M \cap (Q_{w_1} \cap \cdots \cap Q_{w_n})$, where the Q_{w_α} are Segre hypersurfaces associated to M. Assume that every one of Lewy's curves is a chain.

LEMMA. *If $Q_{\bar{w}_0}$ is transverse to M, $S_{\bar{w}_0} = Q_{\bar{w}_0} \cap M$ is a totally geodesic hypersurface in M.*

PROOF. Let $p \in S_{\bar{w}_0}$. Since the map $F: \mathfrak{M} \to \mathbf{P}T^*\mathbf{C}^{n+1}$ is surjective, the $T_pQ_{\bar{w}}$ fill out a neighborhood of $T_pQ_{\bar{w}_0}$, for w near w_0 such that $p \in Q_{\bar{w}}$. Hence every direction tangent to $S_{\bar{w}_0}$ and transverse to the maximal complex tangent space may be described as $T_pM \cap T_pQ_{\bar{w}_0} \cap T_pQ_{\bar{w}_1} \cap \cdots \cap T_pQ_{\bar{w}_{n-1}}$ for some $w_1, \ldots, w_{n-1}$ near w_0. Thus every chain of M tangent to $S_{\bar{w}_0}$ may be written as $M \cap Q_{\bar{w}_0} \cap \cdots \cap Q_{\bar{w}_{n-1}} \subset S_{\bar{w}_0}$, since every one of Lewy's curves is a chain. By Proposition (4.7), $S_{\bar{w}_0}$ is totally geodesic.

As w_0 varies, the tangent spaces $TS_{\bar{w}_0}$ vary over an open set of tangent CR hyperplanes. In particular, the hypotheses of the stronger version of the theorem above (see Remark (2)) are satisfied. Hence, M is flat. We have proven

THEOREM. *Let M be a real-analytic strictly pseudoconvex real hypersurface in $\mathbf{C}^{n+1}$, $n \geqslant 2$. If all of Lewy's curves on M are chains, then M is locally biholomorphically equivalent to the unit sphere in $\mathbf{C}^{n+1}$.*

REFERENCES

1. S. S. Chern, *On the projective structure of a real hypersurface in $\mathbf{C}_{n+1}$*, Math. Scand. **36** (1975), 74–82.

2. S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

3. J. J. Faran, *Segre families and real hypersurfaces*, Thesis, University of California at Berkeley, 1978.

4. B. Segre, I. *Intoruo al problema di Poincaré della rappresentazione pseudo-conforme*, Rend. Acc. Lincei **13** (1931), 676–683; II. *Questioni geometriche legate colla teoria delle funzioni di due variabili complesse*, Rend. Semin. Mat. Roma **7** (1931).

5. S. M. Webster, *Real hypersurfaces in complex space*, University of California at Berkeley, 1975.

6. _____, *On the mapping problem for algebraic real hypersurfaces*, Invent. Math. **43** (1977), 53–68.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540